

Connectivity of Graphs Induced by Directional Antennas

Mirela Damian*

Robin Flatland†

Abstract

This paper addresses the problem of finding an orientation and a minimum radius for directional antennas of a fixed angle placed at the points of a planar set S , that induce a strongly connected communication graph. We consider problem instances in which antenna angles are fixed at 90° and 180° , and establish upper and lower bounds for the minimum radius necessary to guarantee strong connectivity. In the case of 90° angles, we establish a lower bound of 2 and an upper bound of 7. In the case of 180° angles, we establish a lower bound of $\sqrt{3}$ and an upper bound of $1 + \sqrt{3}$. Underlying our results is the assumption that the unit disk graph for S is connected.

1 Introduction

Let S be a set of points in the plane representing wireless nodes. Assume that each node is equipped with one directional antenna, geometrically represented as a wedge with angular aperture α and radius r (see Figure 1a). An antenna orientation is given by the counterclockwise angle θ measured from the positive x -axis to the bisector of the wedge. The *communication graph* $\vec{G}(S)$ formed by the antennas placed at points in S is a directed graph with vertex set S and edges \vec{ab} directed from a to b , if and only if the antenna wedge at a covers b . Let $\text{UDG}(S)$ denote the unit disk graph for S (i.e., the graph in which any two points in S within unit distance are connected by an edge). In this paper we address the following problem.

Let S be a planar point set such that $\text{UDG}(S)$ is connected. For a fixed angle α , find an orientation θ of the antennas at the points in S and a minimum radius r for which the communication graph $\vec{G}(S)$ is *strongly connected*.

We consider instances of this problem for $\alpha = 180^\circ$ (Section 2) and $\alpha = 90^\circ$ (Section 3), and establish lower and upper bounds for the minimum radius required to guarantee strong connectivity. For the case $\alpha = 90^\circ$, we establish a lower bound of 2 and an upper bound of 7. For the case $\alpha = 180^\circ$ angles, we establish a lower bound of $\sqrt{3}$ and an upper bound of $1 + \sqrt{3}$. Underlying these results is the assumption that $\text{UDG}(S)$ is connected. We note that the recent related work by Ben-Moshe et. al [2] also considers 90° -antennas but with orientations restricted to the four standard quadrant directions, and it gives an algorithm for constructing a bidirectional communication graph using a radius value r that is at most twice the optimal value.

Throughout the paper, we use the notation $\text{MST}(S)$ to refer to a minimum spanning tree of S of *maximum degree five*, which can be constructed using the algorithm by Wu et al. [1]. We say that a point $a \in S$ *sees* $b \in S$ if and only if the antenna wedge at a covers b .

*Villanova University, Villanova, USA, mirela.damian@villanova.edu.
Partially supported by NSF grant CCF-0728909.

†Siena College, Loudonville, USA, flatland@siena.edu.

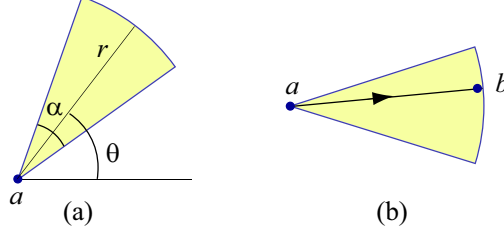


Figure 1: (a) Directional antenna represented as a wedge of angle α and radius r (b) ab is a directed edge in the communication graph.

2 180° Antennas

Theorem 1 *For directional antennas with $\alpha = 180^\circ$, $r \geq \sqrt{3}$ is sometimes necessary to build a strongly connected communication graph.*

Proof. Figure 2 shows a point set for which $r \geq \sqrt{3}$ is necessary. The solid line segments show the UDG; all angles are 120° . Note that for any $r < \sqrt{3}$, any antenna placed at the point labeled p covers exactly two of its neighbors in the UDG and no other points. Split the UDG into two pieces, L and R , by removing the edge connecting p to its uncovered neighbor. Let R be the piece containing p . Observe that for any point $p' \neq p$ in R , the distance from p' to any point in L is at least $\sqrt{3}$. Since messages must flow from R to L , $r \geq \sqrt{3}$ is necessary. ■

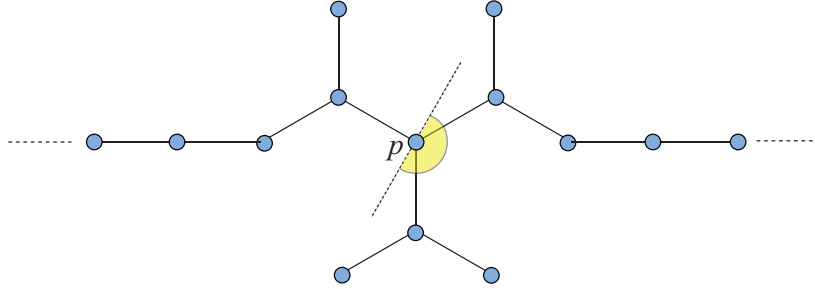


Figure 2: $r \geq \sqrt{3}$ is necessary for this point set when $\alpha = 180^\circ$.

Theorem 2 *For directional antennas with $\alpha = 180^\circ$, $r = 1 + \sqrt{3}$ suffices to build a strongly connected communication graph for a planar point set S .*

Proof. We begin by constructing $\text{MST}(S)$. Let a point in S with a highest y coordinate be the root. We first partition the nodes of $\text{MST}(S)$ into groups, then show inductively how to orient the antennas in each group to form the communication graph. To identify the groups, pick a node of height one and place it in a group along with its children. Conceptually imagine removing this group from the tree, then repeat the process until no nodes are left (with the possible exception of the root). See Figure 3 for an example of node grouping.

We prove the theorem inductively on the number of groups g in $\text{MST}(S)$. The base case corresponds to a tree with one group only ($g = 1$). Let p be the parent node, and c_1 an arbitrary child of p . Orient the antennas at p and c_1 so that they are both aligned with the segment pc_1 and cover opposite sides of the plane. See Figure 4. This placement establishes direct bidirectional

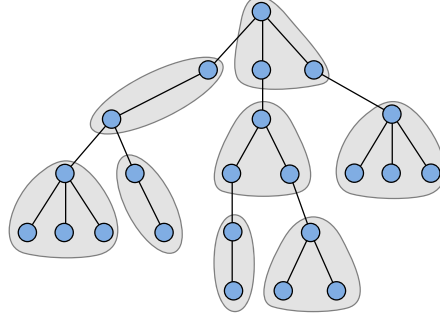


Figure 3: Theorem 1: Tree partitioned into groups.

communication between p and c_1 since the two cones overlap along the segment pc_1 . For the other children (if any), orient their antennas in any direction that includes p . This enables direct communication from each child c to p . Communication from p to c is indirect if c lies outside the antenna wedge at p , in which case the communication path is (p, c_1, c) . Note that the distance from c_1 to any other child of p is at most 2, therefore the radius $r > 2$ claimed by the theorem guarantees connectivity from c_1 to c .

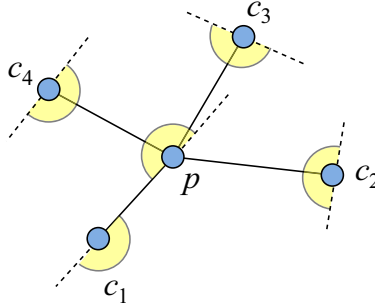


Figure 4: Theorem 1: Orientation of antennas in the base case.

The inductive hypothesis claims that, in the case of a tree $\text{MST}(S)$ composed of g node groups, for some $g \geq 1$, there is an orientation of antennas at the nodes of $\text{MST}(S)$ that satisfies the theorem. In addition, the inductive hypothesis requires that the root of $\text{MST}(S)$ can reach any hop within a unit distance. Note that this is true of the base case, with help from c_1 if the hop is not covered by p 's wedge.

To prove the inductive step, consider a tree $\text{MST}(S)$ with $g + 1$ groups. Let p be the root of $\text{MST}(S)$, and call the group containing p the *root group*. We discuss four cases, depending on the number of nodes in the root group. The antenna placement for each of these cases is depicted in Figure 5. Observe that in the case of a triplet (Figure 5b), p 's antenna is oriented so that it covers both children. It can be verified, in the cases with one and two children depicted in Figure 5(a, b), $r \leq 2$ guarantees strong connectivity, and p can reach any hop within a unit distance.

The cases with the root group composed of four and five nodes follow a similar pattern, once divided into pairs and triplets of nodes, as depicted in Figure 5 (c, d). The dotted lines indicate children that have been paired. Pairs consisting of two children (see c_2, c_3 in Figure 5c) must be carefully selected to achieve $r \leq 1 + \sqrt{3}$; the requirement is that the two paired children form a smallest angle at p . Since the cases under discussion involve three or four children of p , an upper bound on a smallest such angle is 120° . It follows that the distance between the paired children

is at most $\sqrt{3}$. Then $r \leq 2$ guarantees the following: (i) each pair and triplet of nodes is strongly connected, (ii) each node in a pair can send messages to p (because a node in a pair can reach its counterpart node with a setting $r = \sqrt{3}$, and at least one node in a pair can reach p with a setting $r = 1$), and (iii) p can reach any hop within unit distance (which includes its children). It follows that the communication graph for the root group is strongly connected. It remains to show that $\text{MST}(S)$ is strongly connected.

(We pause here to note that a group cannot have five children, since each node has degree at most five in $\text{MST}(S)$, and the parent accounts for one of these degree units. The one exception is the root of the entire tree. But since the root is the point with a highest y coordinate, all its children must lie in a halfplane. The minimum angle separating two adjacent points in a planar minimum spanning tree is 60° , so the root can also have most four children.)

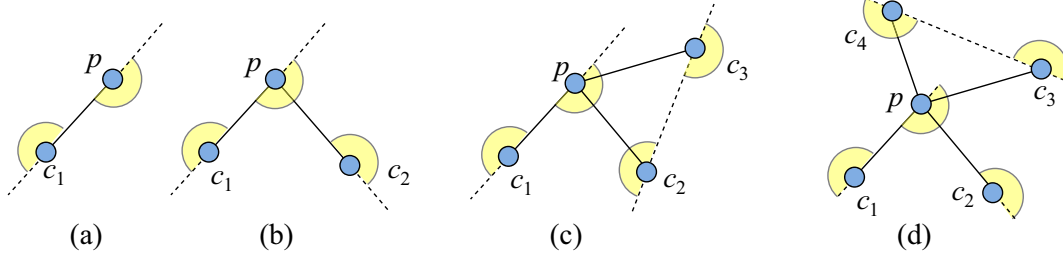


Figure 5: Theorem 1: Orientation of antennas in node groups.

By the inductive hypothesis, each subtree $T \subset \text{MST}(S)$ attached to a node in the root group forms a strongly connected communication graph. In addition, the root of T can reach any hop within unit distance, and therefore it can send messages up to the point of attachment. So to complete the inductive step, we must show that each node in the root group can send messages down to the root of an attached subtree. This is trivially true for p , since p can communicate with any point within unit distance, as established above.

Now consider the case when a child c of p is the attachment point for a subtree with root q . If c plays the role of c_1 in a pair or a triplet (as in Figures 5a,b) and the antenna wedge at c_1 does not cover q , then $r \leq 2$ suffices to establish the communication path (c_1, p, q) . If c plays the role of c_2 in a triplet (as in Figure 5b) and c_2 does not see q , then there are two cases to consider. First, if p sees q , then the communication path is (c_2, p, q) . Otherwise, c_1 must see q . The segment c_2q cannot cross c_1p since minimum spanning tree edges do not cross. This implies that q is confined to the shaded region from Figure 6, therefore the distance from q to c_1 is at most 2. It follows that $r \leq 2$ establishes the communication path (c_2, p, c_1, q) .

Finally, suppose that q is attached to a child of p that was paired with another child of p , such as c_2 in Figure 5c. If c_2 cannot see q , then c_3 must be able to see q , so c_2 can use the communication path (c_2, c_3, q) . Since the distance between c_2 and c_3 is at most $\sqrt{3}$, the last hop from c_3 to q is no longer than $1 + \sqrt{3}$, matching the upper bound on r stated in the theorem.

It is possible that the root group contains a single node, the root p of $\text{MST}(S)$. In this case, we deal with p separately by orienting its antenna in the negative y direction. The root is a highest point and therefore it can see all its children, establishing direct communication with them. By the inductive hypothesis, the children of p can also send messages to p , so this case is settled as well. ■

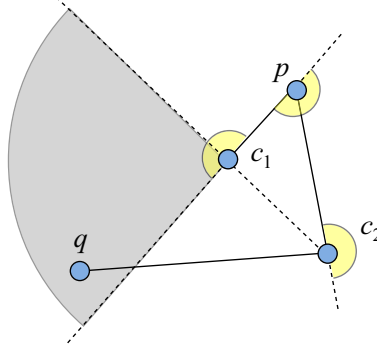


Figure 6: The communication path from c_2 to q is (c_2, p, q) .

3 90° —Antennas

Theorem 3 *For directional antennas with $\alpha = 90^\circ$, $r \geq 2$ is sometimes necessary to build a communication graph.*

Proof. Consider a set of points positioned along the x -axis at unit intervals. An antenna placed at a point p can only cover points to one side, say its left side (so the antenna at p is oriented to the left). To enable messages to flow from points left of p to points right of p , the antenna at some point left of p must be oriented to the right. The nearest such point is p 's left neighbor, which is at distance two from p 's right neighbor. Therefore, $r = 2$ is necessary. ■

Theorem 4 *For any four points in general position, their 90° —antennas can be oriented such that (i) a radius equal to the maximum pairwise distance between the four points guarantees strong connectivity of the four points, and (ii) the four antennas cover the entire plane.*

Proof. Consider first the case when the four points lie in convex position. Let a, b, c, d be the points in clockwise order around the hull. Since $abcd$ is convex, the segments ac and bd must intersect, as illustrated in Figure 7a. Assume without loss of generality that ac is the longer of the two segments, and therefore the projection of at least one of b and d onto the line supported by ac lies on the segment ac . The orientation of antennas depends on the counterclockwise angle β from ac to bd . We will assume $\beta \leq 90^\circ$; the case when $90^\circ < \beta \leq 180^\circ$ is handled symmetrically by reflection about the vertical. It is not difficult to see that the orientation of antennas from Figure 7a covers the entire plane, since ab and cd intersect and β is less than 90° . This settles claim (ii) of the lemma. We now turn to claim (i) of the lemma.

Let each antenna wedge have radius equal to the maximum pairwise distance between a, b, c, d . First note that, for each node pair (a, c) and (b, d) , each node is contained in the antenna wedge of the counterpart node, enabling direct communication between the nodes within a pair. Communication between the pairs is settled as follows. Assume that it is the projection of d that lies on the segment ab , as shown in Figure 7a. Then d is contained in a 's wedge, and d 's wedge contains c , thus enabling full communication between the pairs, as illustrated in Figure 7b.

Consider now the case when the four points do not lie in convex position. Then three of the points, say a, b, c , comprise a triangle that contains the fourth point, d . See Figure 7c. Assume without loss of generality that ac is a longest edge of $\triangle abc$. Then the projection b' of b onto ac lies interior to the segment ac . Let $\triangle abb'$ contain d (the situation when $\triangle cbb'$ contains d is vertically symmetric). The antenna orientation is depicted in Figure 7c: all antenna wedges have

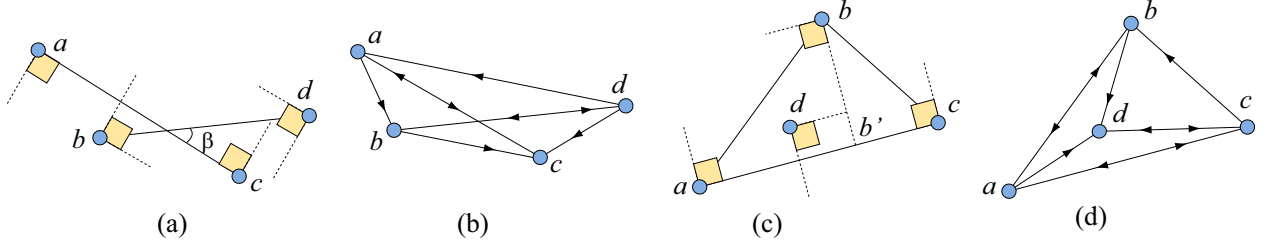


Figure 7: Antennas for points a, b, c, d in convex position (a) and corresponding communication graph (b). Antennas for points a, b, c, d in non-convex position (c) and corresponding communication graph (d).

one boundary line parallel to ac ; the antennas at points a and b face each other, and similarly at points c and d . This guarantees coverage of the entire plane. In terms of connectivity, note that the nodes within each pair (a, b) and (c, d) can communicate directly in both directions. Since $d \in \triangle abb'$, both a and b can see d , and c can see a and b (recall that ac is the longer side of $\triangle abc$, therefore $\angle acb$ is acute, which implies that c sees b). This establishes full communication between the pairs. ■

Theorem 5 *For directional antennas with $\alpha = 90^\circ$, $r = 7$ suffices to build a strongly connected communication graph for a planar point set S .*

Proof. The case when S consists of two points a and b is trivial: orient the antennas at a and b to point to each other. If S consists of three points a, b and c , then $\triangle abc$ has at least two angles strictly smaller than 90° . Orient the antennas at the apexes of these two angles to cover the entire triangle, then orient the third antenna toward either of the other two (see Figure 8a). Then $r = 2$

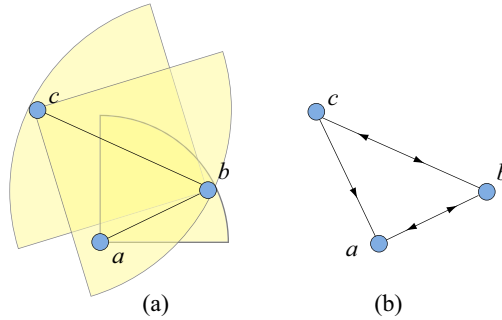


Figure 8: A set S of 3 points. (a) Antenna orientation θ (b) Communication graph.

suffices to form a strongly connected communication graph, since $\max\{|ab|, |ac|, |bc|\} \leq 2$.

We now turn to the general case $|S| \geq 4$. We create groups of nodes in $\text{MST}(S)$ recursively as follows. Starting at the bottom of $\text{MST}(S)$, identify a smallest subtree $T \subseteq \text{MST}(S)$ of four or more nodes, whose removal does not disconnect $\text{MST}(S)$. It can be verified that such a subtree is topologically equivalent to one of the subtrees shown in Figure 9 (note that the dashed connections are possible, but not required in the subtree). Remove T from $\text{MST}(S)$ and recurse. The process stops when $\text{MST}(S)$ is left with three or fewer nodes. The result is a collection \mathcal{C} of node groups, each with four or more vertices, with the possible exception of the root subtree (the one containing the root of $\text{MST}(S)$). In each group we select four representative nodes, one of which must be the

root of the group subtree, and the other three could be arbitrary. For definitiveness we select the nodes shaded in Figure 9 as representative nodes.

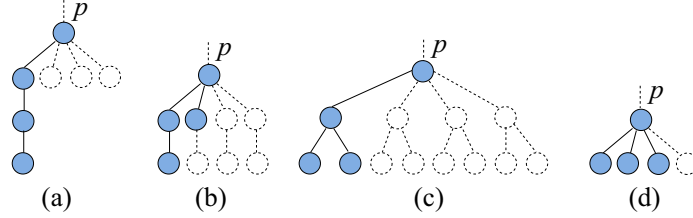


Figure 9: Groups of four nodes that enable the use of Theorem 4. Dashed connections may or may not exist.

We prove the theorem inductively on the number of groups g in $\text{MST}(S)$. The base case with $g = 1$ corresponds to a group of nodes arranged in a subtree topologically equivalent to one of the trees depicted in Figure 9. The representative node set in each group is $R = \{p, a, b, c\}$, with p the root of the group subtree. Note that the maximum pairwise distance between nodes in R is $d_{\max} = 3$ for the subtree depicted in Figure 9a, and $d_{\max} = 2$ for the subtrees depicted in Figure 9(b, c, d). We use Theorem 4 on R to determine an orientation of the antennas at nodes in R that strongly connects R , for $r = d_{\max}$. Then $r = d_{\max} + 1$ enables any node in R to reach any hop within unit distance, because the antennas at nodes in R collectively cover the entire plane.

The inductive hypothesis is that there is an orientation of antennas at the nodes of $\text{MST}(S)$ consisting of g or fewer groups, that satisfies the theorem. In addition, the inductive hypothesis requires that the root of $\text{MST}(S)$ can reach any hop within a unit distance. This additional requirement was already established for the base case.

To prove the inductive step, consider a tree $\text{MST}(S)$ with $g + 1$ groups. Assume first that the root group contains at least four nodes, so they are arranged in a subtree $T \subseteq \text{MST}(S)$ topologically equivalent to one of the trees from Figure 9. As in the base case, we orient the antennas at the representative nodes of T as in Theorem 4, to establish coverage of the plane and strong connectivity between these nodes, for $r = d_{\max}$. For each non-representative node x , orient the antenna at x to point towards a closest representative node y . A simple analysis of the tree topologies from Figure 9 indicate that, in order to establish a connection from x to y , a radius of 1 for the antenna at x suffices for the cases depicted in Figure 9(a,d), and a radius of 2 suffices for the cases depicted in Figure 9(b, c). Summing up these values with d_{\max} , we obtain that $r = 4$ establishes full connectivity between the nodes of T (since one of the nodes in R can reach x in this case as well). We now show that $r = 5$ guarantee strong connectivity of $\text{MST}(S)$.

The inductive hypothesis, along with the fact that each child in $\text{MST}(S)$ is within unit distance from its parent, implies that each subtree attached to a node $x \in T$ can send messages up to x (see the circular arcs in Figure 10a). We have established that a setting of $r = 4$ enables strong connectivity between the nodes of T . It follows that $r = 5$ enables each node $x \in T$ to reach each child y of x , because with this setting at least one of the nodes in R can reach y (since their antennas cover the entire plane), and x can reach any node in R . In addition, $r = 5$ enables the root of $\text{MST}(S)$ to reach any hop within unit distance (by a similar argument). This settles the inductive step for this case.

If the node group at the root of $\text{MST}(S)$ contains fewer than four nodes, this group can be viewed as attached to the root q of any adjacent node group in $\text{MST}(S)$. This idea is illustrated in Figure 10b, where a root subtree of three nodes is attached in turn to each “full” subtree (with four or more nodes). Regardless of the topological structure of the root subtree T , the maximum

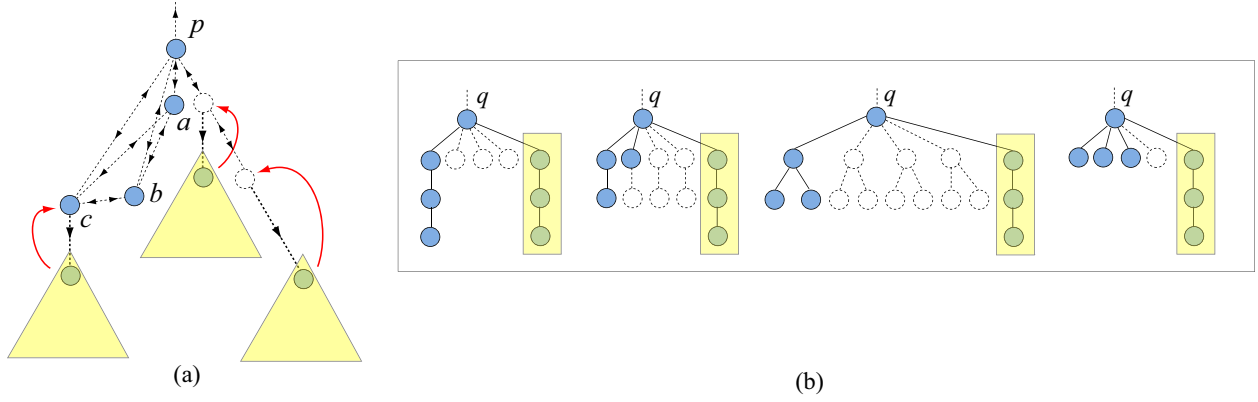


Figure 10: (a) The inductive step for Theorem 5. The communication paths indicated by circular arcs are guaranteed by the inductive hypothesis. The directed dashed edges represent communication paths established in the inductive step. (b) The root subtree with three nodes (boxed) viewed as part of a child subtree.

distance between any two nodes in T does not exceed 2. Orient the antennas at each node in T towards q . Then each node in T can reach q with $r = 3$. A simple analysis of the configurations from Figure 10b shows that a representative node of the subtree rooted at q can reach any node in T with an increase of 2 in its transmission radius. Then the same analysis as before shows that $r = 7$ settles the inductive step. It is likely that a more complex analysis of this special case (with the root group composed of three or fewer nodes) can maintain the previously established bound of $r = 5$. Such an analysis is left for future work. ■

References

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